

# Completely Transitive Designs

Chris D. Godsil and Cheryl E. Praeger

Manuscript from 1997; tidied a bit

## Abstract

We view a design  $\mathcal{D}$  as a set of  $k$ -subsets of a fixed set  $X$  of  $v$  points. A  $k$ -subset of  $X$  is at distance  $i$  from  $\mathcal{D}$  if it intersects some  $k$ -set in  $\mathcal{D}$  in  $k - i$  points, and no subset in more than  $k - i$  points. Thus  $\mathcal{D}$  determines a partition by distance of the  $k$ -subsets of  $X$ . We say  $\mathcal{D}$  is completely transitive if the cells of this partition are the orbits of the automorphism group of  $\mathcal{D}$  in its induced action on the  $k$ -subsets of  $X$ . This paper initiates a study of completely transitive designs  $\mathcal{D}$ . A classification is given of all examples for which the automorphism group is not primitive on  $X$ . In the primitive case the focus is on examples with the property that any two distinct  $k$ -subsets in  $\mathcal{D}$  have at most  $k - 3$  points in common. Here a reduction is given to the case where the automorphism group is 2-transitive on  $X$ . New constructions are given by classifying all examples for some families of 2-transitive groups, leaving several unresolved cases.

## 1 Introduction

A partition  $\pi = C_1, \dots, C_d$  of the vertex set of a graph  $Y$  is *equitable* if, for each  $i$  and  $j$ , the number of neighbours in  $C_j$  of a vertex in  $C_i$  is determined by  $i$  and  $j$ . A typical example is provided by the orbits of any group of automorphisms of  $Y$ . If  $C \subseteq Y$  then the *distance partition* with respect to  $C$  is the partition whose  $i$ -th cell consists of the vertices in  $Y$  at distance  $i$  from  $C$ , that is vertices whose minimum distance from some vertex of  $C$  is  $i$ . The maximum distance of a vertex in  $Y$  from  $C$  is the *covering radius* of  $C$ , and will usually be denoted by  $r$ . The distance partition relative to a subset  $C$  is not usually equitable, but when it is we call  $C$  a *completely regular* subset of

$Y$ . Moreover a completely regular subset  $C$  is said to be *completely transitive* if the cells of the distance partition with respect to  $C$  are the orbits of some group of automorphisms of  $Y$ . Completely regular subsets are important in coding theory and, more generally, in the theory of distance regular graphs. In particular, a regular graph is distance-regular if and only if each vertex in it is a completely regular subset. (For the last claim see the proof of Theorem 2.2 in [5], for general information on distance-regular graphs see [1, 4] and for more on equitable partitions [4].) Distance-transitive graphs form an important and interesting subclass of distance-regular graphs. In the present context we may view these as vertex transitive graphs with the property that each vertex is a completely transitive subset.

The Johnson graph  $J(v, k)$  is defined as follows. Its vertices are the  $k$ -subsets of a fixed subset of  $v$  points and two  $k$ -subsets are adjacent if they have exactly  $k - 1$  points in common. Since  $J(v, k)$  and  $J(v, v - k)$  are isomorphic, we will normally assume that  $2k \leq v$ . The aim of this paper is to study completely transitive subsets of  $J(v, k)$ .

Let  $C$  be a subset of the vertex set  $V(J(v, k))$  of  $J(v, k)$ . We have already defined the covering radius of  $C$ , but there are two further parameters we need. The *minimum distance*  $\delta$  of  $C$  is simply the minimum distance between two vertices of  $C$ . The *strength* of  $C$  is the largest integer  $t$  such that every  $t$ -subset of the underlying  $v$ -set  $X$  lies in the same number of vertices of  $C$ . Thus, if  $t \geq 1$ , then  $C$  is a  $t$ -design on  $X$ , where here we are using  $t$ -design with its usual meaning, that is a collection  $C$  of  $k$ -subsets of  $X$  with the property that each  $t$ -subset of  $X$  lies in the same number of elements of  $C$ . One result of this paper (??) is that if  $G$  is the automorphism group of a completely transitive subset of  $J(v, k)$  with  $\delta \geq 3$  then  $G$  must act 2-transitively on  $X$ , and hence  $C$  has strength at least two; that is  $C$  is a 2-design.

Completely regular subsets of  $J(v, k)$  are discussed at length in [8, 9, 10].

## 2 Examples

We shall describe the important known classes of completely transitive designs in this section. However we begin by making some general observations about completely regular subsets. First we note an unpublished observation due to A. Neumaier.

**2.1 Lemma.** *Let  $C = C_0$  be a subset of a distance-regular graph with distance partition  $C_0, \dots, C_r$ . Then  $C_0$  is completely regular if and only if  $C_r$  is.*

*Proof.* A simple induction argument shows that since the partition  $C_0, \dots, C_r$  is equitable, any vertex in  $C_i$  is joined by a path of length  $r - i$  to a vertex in  $C_r$ , and no shorter such path exists. Hence  $(C_r, \dots, C_1, C_0)$  is the distance partition with respect to  $C_r$  and so  $C_r$  is completely regular. The result now follows.  $\square$

Of course, the same argument shows that  $C_0$  is completely transitive if and only if  $C_r$  is also. It will be convenient to denote  $C_r$  by  $C_{\text{opp}}$ . The first author has shown that, in the case where  $C$  is a subset of  $J(v, k)$ ,  $C$  and  $C_{\text{opp}}$  always have the same strength (see [?????]). Note that in this case  $C_0$  is completely transitive if and only if the set of complements in  $X$  of the  $k$ -sets in  $C_0$  is a completely transitive subset of  $J(v, v - k)$ . Thus we may always assume that  $k \leq v - k$ .

*Example 1* Let  $Y$  be a subset of  $X$  and let  $C_0$  be the set of  $k$ -subsets  $\alpha$  of  $X$  such that  $\alpha \cap Y$  is maximal. (So if  $k \geq |Y|$  then  $C_0$  consists all  $k$ -subsets which contain  $Y$ ; otherwise it is all the  $k$ -subsets of  $Y$ .) Then  $C_{\text{opp}}$  consists of the  $k$ -subsets of  $X$  whose intersection with  $Y$  is minimal.

In the next four examples, we assume that  $\{Y_1, \dots, Y_b\}$  is a partition of  $X$  with  $|Y_i| = a$  for all  $i$ .

*Example 2* Assume  $b = 2$  and  $k \leq a$ . Let  $C_0$  be the set of all  $k$ -sets contained in  $Y_1$  or  $Y_2$ . (In this case  $C_{\text{opp}}$  consists of all  $k$ -sets which meet one of the  $Y_i$  in  $\lfloor k/2 \rfloor$  points and the other in  $\lceil k/2 \rceil$  points.)

*Example 3* Assume  $a = 2$  and let  $C_0$  be the set of all  $k$ -sets containing at most one element from each  $Y_i$ . (In this case  $C_{\text{opp}}$  consists of the  $k$ -sets which meet at most one of the  $Y_i$  in a single point.)

*Example 4* Assume  $a, b \geq 3$  and  $k = 3$ . Let  $C_0$  be the set of triples meeting each  $Y_i$  in at most one point.

*Example 5* Assume  $k = 2$  and let  $C_0$  be the set of all pairs meeting each  $Y_i$  in at most one point.

In each of Examples 2–5 the automorphism group of  $C_0$  (or  $C_{\text{opp}}$ ) is  $\text{Sym}(a) \wr \text{Sym}(b)$ . In Example 1 the automorphism group is intransitive. We will prove that if  $C_0$  is intransitive and its automorphism group is intransitive, or transitive but imprimitive, then  $C_0$  is one of the above examples.

Next we mention a number of sporadic examples. The set of lines of a projective plane of order two is a completely transitive subset of  $J(7, 3)$  with covering radius one, while the set of lines of a plane of order three is completely transitive in  $J(13, 4)$ . Martin [8] shows that no other projective plane is completely regular. Delsarte [3] observed that the Witt design on 24 points is completely regular, with covering radius two. From Table 1 in [7] we see that  $M_{24}$  has three orbits on sets of size eight, hence  $W_{24}$  is completely transitive. Martin [8] proved that the Witt design on 23 points is completely regular with covering radius three. From Table 1 in [7] we see that  $M_{23}$  has four orbits on sets of size seven, whence the design is completely transitive. By [8] the Witt design on 22 points is not even completely regular (Martin [8, 10]).

### 3 The Intransitive and Imprimitive Cases

Our first result is a characterisation of the completely transitive designs for which the automorphism group  $G$  is intransitive on  $X$ . A group of automorphisms of  $J(v, k)$  will be said to be *completely transitive* if its orbits on  $k$ -subsets form the distance partition for some subset of  $J(v, k)$ .

**3.1 Lemma.** *Let  $C$  be a completely transitive design on a  $v$ -set  $X$  with  $0 < k < v$  admitting a completely transitive group which is intransitive on  $X$ . Then  $C$  is as in Example 1 for some non-empty proper subset  $Y$  of  $X$  and  $\text{Aut}(C) \cong \text{Sym}(Y) \times \text{Sym}(\bar{Y})$ . Any completely transitive group with  $C_0$  as an orbit is transitive on  $\binom{Y}{j} \times \binom{\bar{Y}}{k-j}$  for each  $j$  such that  $0 \leq j \leq \min\{k, |Y|\}$ .*

*Proof.* Let  $C_0, \dots, C_r$  be the cells of the distance partition of  $C$  and let  $G \leq \text{Sym}(X)$  be a completely transitive group preserving  $C$  and intransitive on  $X$ . If  $Y$  is an orbit of  $G$  on  $X$  then any two  $k$ -sets in the same cell  $C_i$  must meet  $Y$  in the same number of points. Suppose  $\alpha \in C$  and  $Y$  is an orbit of  $\text{Aut}(C)$  such that  $|\alpha \cap Y| = i$ , for some positive integer  $i$ . Suppose further that we have elements of  $y, y', z$  and  $z'$  of  $X$  such that

$$y \in \alpha \cap Y, \quad y' \in Y \setminus \alpha, \quad z \in \alpha \setminus Y, \quad z' \in \bar{Y} \setminus \alpha.$$

Then the  $k$ -sets  $(\alpha \setminus \{z\}) \cup \{y'\}$  and  $(\alpha \setminus \{y\}) \cup \{z'\}$  are both adjacent to  $\alpha$  in  $J(v, k)$ , but meet  $Y$  in  $i + 1$  and  $i - 1$  points respectively. Therefore neither of these sets belongs to  $C_0$  and they cannot both lie in  $C_1$ . Thus we conclude that one of the following holds:

- (i)  $Y \subseteq \alpha$ ,
- (ii)  $\alpha \subseteq Y$ ,
- (iii)  $\bar{Y} \subseteq \alpha$ .

In the second case we may replace the  $k$ -sets on  $C_0$  by their complements in  $X$ , thus reducing to case (i). In the third case, we may replace  $Y$  by an orbit of  $\text{Aut}(C)$  contained in  $\bar{Y}$ , with the same result.

Suppose now that (i) is true and that  $|Y| = m$ . We aim to show that  $C_i$  consists of all  $k$ -subsets of  $Y$  which meet  $Y$  in  $m - i$  points. Since  $k < v$  we may choose a point  $z$  not in  $\alpha$  and a point  $y$  in  $Y$ . Then  $(\alpha \setminus \{y\}) \cup \{z\}$  is adjacent to  $\alpha$  in  $J(v, k)$  and meets  $Y$  in  $m - 1$  points. It follows that this  $k$ -set is in  $C_1$ , and hence that all elements of  $C_1$  meet  $Y$  in  $m - 1$  points. Thus if  $x \in \alpha \setminus Y$  and  $z \notin \alpha$  then  $(\alpha \setminus \{x\}) \cup \{z\}$  must lie in  $C_0$  and accordingly  $C_0$  consists of all the  $k$ -subsets of  $X$  which contain  $Y$ . A  $k$ -set which contains exactly  $m - i$  points from  $Y$  meets each element of  $C_0$  in at most  $k - i$  points and hence is at distance at least  $i$  in  $J(v, k)$  from  $C_0$ . From this it follows easily that  $C_i$  consists of all  $k$ -subsets of  $X$  which meet  $Y$  in  $m - i$  points, as asserted, and that  $\text{Aut}(C)$  is transitive on  $\binom{Y}{j} \times \binom{\bar{Y}}{k-j}$  whenever  $0 \leq i \leq m$ . Clearly this will also hold for any subgroup of  $\text{Aut}(C)$  which is transitive on each of the sets  $C_i$ .  $\square$

Next we shall classify all completely transitive designs admitting a transitive imprimitive group  $G$ .

**3.2 Lemma.** *Let  $C$  be a subset of  $V(J(v, k))$ , with  $k \leq v/2$  and suppose that some transitive imprimitive subgroup  $G$  of  $\text{Sym}(X)$  acts completely transitively on  $C_0$ . Then  $C$  is one of Examples 2-5, or its opposite.*

*Proof.* Let  $Y = \{Y_1, \dots, Y_b\}$  be a non-trivial partition of  $X$  preserved by  $G$ , where  $|Y_i| = a$  for all  $i$  and  $v = ab$ . Let  $C_0, \dots, C_r$  be the distance partition of  $C$ , where  $C_0 = C$ . Let  $\alpha$  be a  $k$ -subset of  $X$  and for each  $i = 1, \dots, b$  set  $\alpha_i = \alpha \cap Y_i$  and  $e_i = |\alpha_i|$ . Let  $\mathbf{e}(\alpha)$  be the multiset  $\{e_1, \dots, e_b\}$ . If  $\alpha$  and  $\beta$  lie in the same cell of the distance partition of  $C$  then  $\mathbf{e}(\alpha) = \mathbf{e}(\beta)$ .

We first consider the case  $b = 2$ . Suppose that  $\alpha$  from  $C$  meets both  $Y_1$  and  $Y_2$  and that

$$|\alpha \cap Y_1| \geq |\alpha \cap Y_2|.$$

If  $\alpha \supseteq Y_1$  then, since  $k \leq v/2 = a$ , it follows that  $\alpha = Y_1$  and  $C_0 = \{Y_1, Y_2\}$ . (This is an uninteresting special case of Example 2.) So we may assume that

there are points  $x \in \alpha \cap Y_1$ ,  $y_1 \in Y_1 \setminus \alpha$  and  $y_2 \in Y_2 \setminus \alpha$ . Define  $k$ -subsets of  $X$  by

$$\beta_1 := (\alpha \setminus \{x\}) \cup \{y_1\}, \quad \beta_2 := (\alpha \setminus \{x\}) \cup \{y_2\}.$$

Then  $\beta_1$  and  $\beta_2$  are both adjacent to  $\alpha$  in  $J(v, k)$  but  $\mathbf{e}(\alpha) = \mathbf{e}(\beta_1)$ , and  $\mathbf{e}(\alpha) \neq \mathbf{e}(\beta_2)$  unless  $|\alpha \cap Y_1| = (k+1)/2$ . Suppose first that  $\alpha \subset Y_1$ . Then  $\beta_2 \in C_1$  and  $\beta_1 \in C_0$ . It follows that all  $k$ -subsets of  $Y_1$  lie in  $C_0$ ; so that  $C = C_0$  consists of all  $k$ -subsets of  $Y_1$  or  $Y_2$ , i.e, it is as in Example 2. Now suppose that  $\alpha \not\subset Y_1$ . Then  $k > i := |\alpha \cap Y_1| \geq |\alpha \cap Y_2| = k - i > 0$ . Thus there is a point  $x_2 \in \alpha \cap Y_2$  and the  $k$ -subset  $\beta_3 := (\alpha \setminus \{x_2\}) \cup \{y_1\}$  is adjacent to  $\alpha$  in  $J(v, k)$ . Moreover  $\mathbf{e}(\beta_3) \neq \mathbf{e}(\alpha)$ , so  $\beta_3 \in C_1$ . So now we have  $\mathbf{e}(\beta_2) \neq \mathbf{e}(\beta_3)$  (since  $i \geq k/2$ ) and  $\mathbf{e}(\beta_1) \neq \mathbf{e}(\beta_3)$ , and hence  $\beta_1, \beta_2 \notin C_1$ . Since  $\beta_1, \beta_2$  are both adjacent to  $\alpha$  it follows that  $\beta_1, \beta_2 \in C_0$  whence  $\mathbf{e}(\beta_2) = \mathbf{e}(\alpha)$ . Therefore  $i = (k+1)/2$ , and  $C_0$  consists of all  $k$ -subsets which contain  $(k+1)/2$  points of one of the  $Y_j$  and  $(k-1)/2$  points of the other. In this case  $C_{\text{opp}}$  consists of all  $k$ -subsets of  $Y_1$  or  $Y_2$  and so  $C_{\text{opp}}$  is as in Example 2.

Next consider the case  $a = 2$ . Call a subset of  $X$  which meets each set  $Y_i$  in at most one point a *partial transversal*. Suppose  $\alpha \in C$  meets at least two of the sets  $Y_i$  in one point, but is not a partial transversal. Then we may assume that  $\alpha$  is disjoint from  $Y_1$ , meets both  $Y_2$  and  $Y_3$  in one point, and contains  $Y_4$ . Let  $\beta_1$  be the  $k$ -set obtained from  $\alpha$  by deleting one of the points in  $Y_4$  and replacing it with a point in  $Y_1$ . Let  $\beta_2$  be the  $k$ -set obtained by deleting the point in  $Y_2$  from  $\alpha$  and replacing it with the point in  $Y_3 \setminus \alpha$ . Finally let  $\beta_3$  be obtained by removing one of the points in  $Y_4$  and adding the missing point from  $Y_3$ . Then  $\mathbf{e}(\beta_1)$ ,  $\mathbf{e}(\beta_2)$  and  $\mathbf{e}(\beta_3)$  are all distinct. But if  $\alpha \in C_0$  then all its neighbours must lie in  $C_0 \cup C_1$ . Thus we conclude that either  $\alpha$  is a partial transversal or it must meet at most one of the  $Y_i$  in a single point.

Assume first that  $\alpha$  from  $C_0$  is a partial transversal. Then all elements of  $C_0$  are partial transversals. The neighbours of  $\alpha$  in  $J(v, k)$  are either partial transversals or contain exactly one of the sets  $Y_i$ . From this it follows  $C_0$  is set of all partial transversals of size  $k$ , and so  $C_0$  is as in Example 3. Thus we are left with the possibility that  $\alpha$  meets at most one  $Y_i$  in a single point, and thus all elements of  $C_0$  contain exactly  $\lfloor k/2 \rfloor$  of the sets  $Y_i$ . In this case  $C_{\text{opp}}$  must contain a partial transversal and so, by what we have just proved,  $C_0$  must be the opposite of a completely transitive subset as in Example 3.

If  $k = 2$  or  $3$  then it is easy to see that  $C_0$  is as in Example 4 or Example 5 respectively. Accordingly we may assume that  $a \geq 3$ ,  $b \geq 3$  and  $k \geq 4$ . Let

$k_1$  and  $a_1$  be respectively the quotient and remainder when  $k$  is divided by  $a$ . Then there is a  $k$ -subset  $\alpha$  of  $X$  such that  $k_1$  of the entries of  $\mathbf{e}(\alpha)$  are equal to  $a$ , one is  $a_1$  and the rest are all zero. Note that since  $v = ab$  and  $k \leq v/2$ ,

$$k_1 = \lfloor \frac{k}{a} \rfloor \leq \lfloor \frac{b}{2} \rfloor \quad (3.1)$$

and consequently there must be at least one entry of  $\mathbf{e}(\alpha)$  equal to zero. Assume for the moment that  $k_1 > 0$  and  $0 < a_1 < a - 1$ . Then there is a  $k$ -subset  $\beta$  adjacent to  $\alpha$  with  $k_1 - 1$  of the entries of  $\mathbf{e}(\beta)$  equal to  $a$ , one entry equal to  $a - 1$ , another equal to  $a_1 + 1$ , and the remaining entries zero. There is also a  $k$ -subset  $\gamma$  adjacent to both  $\alpha$  and  $\beta$  with  $k_1 - 1$  of the entries of  $\mathbf{e}(\gamma)$  equal to  $a$ , one entry equal to each of  $a - 1$ ,  $a_1$  and 1, and the rest zero. Suppose that  $\alpha \in C_i$  for some  $i$ . Since  $\mathbf{e}(\alpha)$ ,  $\mathbf{e}(\beta)$  and  $\mathbf{e}(\gamma)$  are all different, one of  $\beta$  and  $\gamma$  must lie in  $C_{i-1}$  and the other in  $C_{i+1}$ . As  $\beta$  and  $\gamma$  are adjacent, this is impossible.

We are left with the possibilities that  $k_1 = 0$  or that  $a_1 = 0$  or  $a - 1$ . An obvious analogue of the argument of the last paragraph will work if we have a  $k$ -subset  $\alpha$  such that  $\mathbf{e}(\alpha)$  has at least one entry equal to each of  $a$ ,  $x$  and 0, where  $a - 2 \geq x > 0$ . Suppose first that  $a_1 = 0$ . Then  $0 < k_1 \leq b - 2$ , by (3.1), and it follows that there is a  $k$ -subset  $\alpha$  such that  $\mathbf{e}(\alpha)$  has  $k_1 - 1$  entries equal to  $a$ , one entry equal to each of  $a - 1$  and 1, and the rest zero. If  $k_1 \geq 2$  then by (3.1),  $b \geq k_1 + 2$ , and we may use  $\alpha$  in the above argument (with  $x = 1$ ). On the other hand if  $k_1 = 1$  so that  $k = a$ , then there are mutually adjacent  $k$ -subsets  $\alpha, \beta, \gamma$  with  $\mathbf{e}(\alpha) = \{a - 1, 1, 0^{b-2}\}$ ,  $\mathbf{e}(\beta) = \{a - 2, 2, 0^{b-2}\}$ , and  $\mathbf{e}(\gamma) = \{a - 2, 1, 1, 0^{b-3}\}$ . Next suppose that  $a_1 = a - 1$ . In this case if  $k_1 > 0$  then there is a  $k$ -subset  $\alpha$  with  $k_1$  of the entries of  $\mathbf{e}(\alpha)$  equal to  $a$ , one equal to each of  $a - 2$  and 1, and the rest zero. We may use  $\alpha$  in the above argument (with  $x = a - 2$ ) provided that  $b > k_1 + 2$ , while if  $b \leq k_1 + 2$  then by (3.1),  $b$  is 3 or 4 and  $k = (k_1 + 1)a - 1 \geq (b - 1)a - 1 > v/2$  which is not allowed. This leaves the case  $k_1 = 0$ ,  $k = a_1$ . In this case there are mutually adjacent  $k$ -subsets  $\alpha, \beta, \gamma$  with  $\mathbf{e}(\alpha) = \{k - 1, 1, 0^{b-2}\}$ ,  $\mathbf{e}(\beta) = \{k - 2, 2, 0^{b-2}\}$ , and  $\mathbf{e}(\gamma) = \{k - 2, 1, 1, 0^{b-3}\}$ .  $\square$

The problem of classifying completely transitive designs admitting primitive groups is open, but by the results of this section is reduced to the following problem.

**3.3 Problem.** *Classify all completely transitive designs on a  $v$ -set  $X$  admitting a group  $G \leq \text{Sym}(X)$  which is primitive on  $X$ .*

## 4 Designs with Minimum Distance at Least Three

As in the previous section  $C$  is a completely transitive design in  $J(v, k)$ . If  $|C| = 1$  then it is a case of Example 1.1 with  $\ell = k$ , so assume that  $|C| \geq 2$ . Let  $\delta$  denote the minimum distance between two  $k$ -subsets in  $C$ . The design of Examples 2.1 has  $\delta = 1$ , hence Proposition 3.1 yields that the automorphism group of a completely transitive design with  $\delta \geq 2$  must be transitive on the underlying set  $X$ . Most of the designs in Examples 2.2–5 have  $\delta = 1$ . We describe the exceptions explicitly. The first arises when  $v = 2k$ , when we may take  $C$  to consist of two disjoint  $k$ -subsets of  $X$ . (This is a special case of Example 2.2, and has  $\delta = k$ .) If  $v$  and  $k$  are even and  $C$  consists of the  $k$ -subsets formed by the union of any  $k/2$  of the cells of a fixed partition of  $X$  into pairs, then  $C$  is completely transitive with  $\delta = 2$ . If  $v$  is divisible by three then the  $v/3$  triples in a fixed partition of  $X$  into 3-sets is a completely transitive design with  $k = \delta = 3$ .

Meyerowitz [11] shows that the completely regular subsets of  $J(v, k)$  with strength zero are precisely the designs of Example 2.1 (and their opposites). We note that Martin [8, 9] has shown that a completely regular subset of  $J(v, k)$  with strength one and  $\delta \geq 2$  must be one of the designs just described above. Consequently any completely regular subset of  $J(v, k)$  with  $\delta \geq 3$  must have strength at least two, and is therefore a 2-design in the usual sense of the word. From now on we concentrate on the case  $\delta \geq 3$ .

**4.1 Theorem.** *If  $C$  is a completely transitive design with  $|C| \geq 2$  and  $\delta \geq 3$  and  $\alpha \in C$  then  $\text{Aut}(C)_\alpha$  is transitive on the Cartesian product  $\alpha \times (X \setminus \alpha)$ . Further, either:*

- (a)  $v = 2k \geq 6$  and  $C$  consists of two disjoint  $k$ -subsets,
- (b)  $v = 3b$  and  $C$  consists of  $b$  pairwise disjoint triples, or
- (c)  $\text{Aut}(C)$  is 2-transitive on  $X$ .

*Proof.* Let  $C_0, \dots, C_r$  be the distance partition of  $C$ , where  $C_0 = C$ , and assume  $\alpha \in C_0$ . Assume  $x, x' \in \alpha$  and  $y, y' \in X \setminus \alpha$ . Then  $\beta = (\alpha \setminus \{x\}) \cup \{y\}$  and  $\beta' = (\alpha \setminus \{x'\}) \cup \{y'\}$  are both adjacent to  $\alpha$  and, as  $\delta \geq 3$ , both  $\beta$  and  $\beta'$  lie in  $C_1$ . Hence  $\beta^g = \beta'$  for some  $g \in G$ . If  $\alpha^g \neq \alpha$  then we  $\alpha^g \in C_0^g = C_0$  and so  $\alpha$  and  $\alpha^g$  are at distance at most two in  $J(v, k)$ , which is a contradiction.



Hence  $\alpha^g = \alpha$ , so  $g \in G_\alpha$  and  $g$  maps  $x$  to  $x'$  and  $y$  to  $y'$ . Further, if we take  $x = x'$  above then we see that  $\text{Aut}(C)_{\alpha,x}$  is transitive on  $X \setminus \alpha$ , whence  $\text{Aut}(C)_\alpha$  is transitive on  $\alpha \times (X \setminus \alpha)$ .

By Lemma 3.1 either  $C$  is as in the statement or  $\text{Aut}(C)$  is primitive on  $X$ . Assume the latter holds, and let  $B(x)$  be the intersection of all  $k$ -subsets of  $C_0$  which contain  $x$ . It is easy to see that  $B(x)$  is a block of imprimitivity for  $\text{Aut}(C)$  in  $X$  containing  $x$ , and as  $\text{Aut}(C)$  is primitive we have  $B(x) = \{x\}$ . Let  $y, y'$  be distinct points of  $X \setminus \{x\}$ . Then there are  $\alpha, \alpha' \in C_0$  containing  $x$ , such that  $y \notin \alpha$  and  $y' \notin \alpha'$ . Now  $X \setminus \alpha$  and  $X \setminus \alpha'$  have a point in common,  $z$  say, since  $k \leq v/2$  and  $x \in \alpha \cap \alpha'$ . Since  $\text{Aut}(C)_{\alpha,x}$  and  $\text{Aut}(C)_{\alpha',x}$  are transitive on  $X \setminus \alpha$  and  $X \setminus \alpha'$  respectively there are elements  $g$  in  $\text{Aut}(C)_{\alpha,x}$  and  $g'$  in  $\text{Aut}(C)_{\alpha',x}$  such that  $y^g = z$  and  $z^{g'} = y'$ . It follows that  $\text{Aut}(C)_x$  is transitive on  $X \setminus \{x\}$ , that is  $\text{Aut}(C)$  is 2-transitive on  $X$ .  $\square$

Next we deal with the case where  $k$  is small.

**4.2 Theorem.** *Let  $C$  be a completely transitive design on a  $v$ -set  $X$  with  $k \leq v/2$  such that  $|C| \geq 2$ ,  $\delta \geq 3$  and  $G$  is 2-transitive. Then either*

- (a)  $v = 13$ ,  $k = 4$ ,  $r = 2$ , and  $C_0$  is the set of lines of the Pappian projective plane  $PG_2(3)$ , or
- (b)  $k \geq 5$  and  $r \leq k - 2$ .

*Proof.* Since  $\text{Aut}(C)$  is 2-transitive on  $X$ , for any  $k$ -subset  $\beta$  and any pair of points  $\{y, y'\} \subseteq \beta$ , there is some  $\alpha \in C_0$  containing  $\{y, y'\}$ . Hence  $\alpha$  and  $\beta$  are distance at most  $k - 2$  in  $J(v, k)$  whence  $\beta \in C_i$  for some  $i \leq k - 2$ . Hence  $r \leq k - 2$ . Since  $\delta \geq 3$ , we have  $k \geq \delta \geq 3$ , and  $r \geq 1$ . If  $k = 3$  then, for  $\beta = \{x, y, z\} \in C_1$ , there are points  $s$  and  $t$  in  $X \setminus \beta$  such that  $\alpha = \{x, y, s\}$  and  $\alpha' = \{x, z, t\}$  belong to  $C_0$  (since  $G$  is 2-transitive on  $X$ ); then  $\alpha$  and  $\alpha'$  are at distance two in  $J(v, k)$  contradicting  $\delta \geq 3$ . Hence  $k \geq 4$ .

Let  $k = 4$ . If  $\alpha = \{x, y, z, w\}$  and  $\alpha' = \{x, y, z', w'\}$  are distinct 4-subsets in  $C_0$  both containing  $x$  and  $y$  then  $\alpha$  and  $\alpha'$  are at distance at most two in  $J(v, k)$ , contradicting  $\delta \geq 3$ . Hence each pair from  $X$  lies in a unique 4-subset in  $C_0$ . So  $C_0$  is the set of blocks of a 2-transitive  $2 - (v, 4, 1)$  design. There are  $\binom{v-2}{2}$  subsets of size four containing  $x$  and  $y$ , one of which is in  $C_0$  and  $2(v-2)$  of which are in  $C_1$ . Since  $v \geq 2k = 8$  it follows that  $\binom{v-2}{2} > 2(v-2) + 1$ , so  $C_2 \neq \emptyset$ . As any 4-subset contains at least two points of some 4-subset in  $C_0$  it follows that  $r = 2$ , and  $\text{Aut}(C)$  is transitive on independent 4-subsets, that is 4-subsets in which no three points lie in a block of  $C_0$ .

By [6], the 2-transitive  $2 - (v, 4, 1)$  designs are the projective spaces  $PG_d(3)$  and affine spaces  $AG_d(4)$  with  $d \geq 2$ , and the Hermitian and Ree unitals on 28 points. Since  $PGL_3(3)$  is transitive on independent 4-subsets of points, and also on triples  $(l, x, y)$ , where the point  $x$  lies on the line  $l$  and the point  $y$  does not, it follows that taking  $C_0$  to be the set of lines of  $PG_2(3)$  gives an example. If  $d \geq 3$  then independent 4-subsets might span a plane or a three-dimensional projective space, so there are no further examples from  $PG_d(3)$ . Similarly for  $AG_d(4)$ , if  $d \geq 3$  then independent 4-subsets might span a plane or a three-dimensional affine space, so  $d = 2$ . But for  $AG_2(4)$ ,  $|C_2| = 2^3 \cdot 3 \cdot 5 \cdot 7$ , and as 7 does not divide  $|AGL_2(4)|$ ,  $\text{Aut}(C)$  cannot be transitive on  $C_2$ . For the Hermitian and Ree unitals,  $|C_0| = 63$ ,  $|C_1| = |C_0| \cdot 4 \cdot 24$ , and hence  $|C_2| = \binom{28}{4} - 97 \cdot |C_0|$ , which is greater than  $|\text{Aut}(C)|$  in either case, so  $\text{Aut}(C)$  cannot be transitive on  $C_2$ .  $\square$

**4.3 Corollary.** *If  $r \leq k - 2$  then  $|\text{Aut}(C)| \geq \frac{1}{k-1} \binom{v}{k}$ . Unless  $k = 6$  and  $v = 12$ , this is at least  $\frac{1}{4} \binom{v}{5}$ .*

*Proof.* The group  $\text{Aut}(C)$  has at most  $k - 1$  orbits on  $k$ -subsets from  $X$ , so some  $\text{Aut}(C)$ -orbit on  $k$ -subsets has length at least  $\frac{1}{k-1} \binom{v}{k}$ . A simple arithmetic computation shows that this is at least  $\frac{1}{4} \binom{v}{5}$  unless  $k = 6, v = 12$ .  $\square$

In fact if  $k < v/2$  then we have

$$|\text{Aut}(C)| \geq \frac{1}{k-1} \binom{v}{k} \geq \frac{1}{k-2} \binom{v}{k-1} \geq \cdots \geq \frac{1}{4} \binom{v}{5}.$$

Applying Corollary 4.3 to some of the known 2-transitive groups is surprisingly effective. We prove the following Lemma as a sample result.

**4.4 Lemma.** *If  $r \leq k - 2$  then the socle  $T$  of  $G$  is not one of  $L_2(q)$ ,  $Sz(q)$ ,  $U_3(q)$ ,  $Ree(q)$ , for any  $q$ .*

*Proof.* We deal with the four families separately.

**Suzuki groups:**  $Sz(q) \leq G \leq \text{Aut } Sz(q)$ ,  $q = 2^{2s+1}$  odd,  $s \geq 1$ . Here  $\frac{1}{4} \binom{v}{5} \leq |\text{Aut } Sz(q)|$  becomes

$$\frac{1}{4} \binom{q^2 + 1}{5} \leq (q^2 + 1)q^2(q - 1)a$$

which implies  $q^5 < (q + 1)(q^2 - 2)(q^2 - 3) \leq 480a$ , which is not true.

**Unitary groups:**  $U_3(q) \leq G \leq \text{PGU}_3(q)$ , with  $q = p^a$  for some prime  $p$ , and  $v = q^3 + 1$ . Here the inequality is

$$\frac{1}{4} \binom{q^3 + 1}{5} \leq (q^3 + 1) \cdot q^3 \cdot (q^2 - 1) \cdot 2a$$

which implies  $(q - 1)^7 < \frac{(q^3 - 1)(q^3 - 2)(q^3 - 3)}{(q^2 - 1)} \leq 960a$ . The inequality  $(q - 1)^7 < 960$  implies  $a = 1, q = 2$  or  $3$ . The exact inequality above holds only for  $q = 2$ , but then  $k \geq 5 > v/2$ .

**Ree groups:**  $\text{Ree}(q) \leq G \leq \text{Aut}(\text{Ree}(q))$ ,  $q = 3^{2s+1}$  and  $v = q^3 + 1$ . The arithmetic is similar to and easier than that for the unitary groups and shows that the inequality is never satisfied.

**1-dimensional linear groups:**  $L_2(q) \leq G \leq \text{P}\Gamma L_2(q)$ ,  $q = p^a$  for some prime  $p$  and  $v = q + 1 \geq 2k \geq 10$ . If  $q = 9$  then  $k = 5$  and by Theorem 4.1,  $25 = k(v - k)$  divides  $|G|$  which is not the case. Similarly if  $q = 11$  then  $k$  is five or six and  $k(v - k)$  does not divide  $|G_\alpha|$ . Hence  $v = q + 1 \geq 14$  and by Corollary 4.3,

$$\frac{1}{4} \binom{q + 1}{5} \leq (q + 1)q(q - 1)a$$

which implies  $(q - 2)(q - 3) \leq 480a$ , whence  $a \leq 5$ . We find from this that  $q \in \{32, 27, 25, 23, 19, 17, 16, 13\}$ . If  $k = 5$  then using the fact that  $5(v - 5)$  divides  $|G|$  we obtain  $q = 16$  as the only possibility. Then the 5-subsets  $\alpha$  in  $C_0$  are orbits of subgroups  $Z_5$  of  $L_2(16)$ , and all such orbits form a single  $L_2(16)$ -orbit on 5-subsets. Thus  $C_0$  is uniquely determined by  $L_2(16)$ , and it follows that  $C_0$  is the set of blocks (circles) of the Miquelian inversive plane of order four, which is a  $3 - (17, 5, 1)$  design. Since any triple of points lies on a unique block in  $C_0$ , it follows that any two 5-subsets of points are at distance at most 2, and hence that  $r = 2$ . However, we have in this case that  $|C_0| = 68, |C_1| = |C_0|.60$ , and hence that  $|C_2| = \binom{17}{5} - |C_0| - |C_1| = 2^2.11^2.17$ , which does not divide  $|G|$ , and so  $G$  is not transitive on  $C_2$ . Hence  $k \geq 6$ . Thus we have  $\frac{1}{5} \binom{q+1}{6} \leq |G|$  whence  $(q-2)(q-3)(q-4) \leq 3600a$ , which implies  $q \in \{17, 16, 13\}$ . Then as  $k(v - k)$  divides  $|G|$  we must have  $q = 17, k = 6$ ; but as three does not divide  $|G_x|$  it is not possible for  $G_{\alpha,x}$  to be transitive on  $X \setminus \alpha$  where  $x \in \alpha \in C_0$ .

Next we partially analyse the situation for 2-transitive groups which do not fit into any infinite family of 2-transitive groups, namely the Mathieu groups,  $L_2(11)$  of degree 11, and the Higman-Sims and Conway groups  $HS$  and  $Co_2$ .

**Mathieu groups**  $M_v$  where  $v \in \{11, 12, 22, 23, 24\}$ , or  $L_2(11)$  with  $v = 11$ , or  $M_{11}$  with  $v = 12$  : Since  $k(v - k)$  divides  $|G_\alpha|$  we have the following possibilities:

- (i)  $v = 11, k = 5, G = M_{11}$  or  $L_2(11)$ ,
- (ii)  $v = 12, k = 6, G = M_{12}$  or  $M_{11}$ ,
- (iii)  $v = 22, 6 \leq k \leq 10, k \neq 9, G = M_{22}$  or  $\text{Aut}(M_{22})$
- (iv)  $v = 23, 5 \leq k \leq 11, k \neq 6, 10, G = M_{23}$
- (v)  $v = 24, 6 \leq k \leq 12, k \neq 7, 11, G = M_{24}$ .

We consider the cases separately. In case (i), if  $G = M_{11}$ , then  $G$  has two orbits on 5-subsets of points and we obtain two completely transitive designs, namely the Witt design on 11 points and its opposite, both preserved by  $M_{11}$ . However in this case,  $\delta = 2$  :-(. If  $G = L_2(11)$  then  $G$  has 4 orbits on 5-subsets of points. Let  $C_0$  be the set of blocks of the  $2 - (11, 5, 2)$  design preserved by  $G$ . Then this is one of the orbits. Since each pair of blocks in  $C_0$  intersect in 2 points,  $\delta = 3$ . If  $\alpha \in C_0$  then, since  $G_\alpha$  is transitive on  $\alpha \times (X \setminus \alpha)$ , it follows that  $G$  is transitive on the 330 subsets in  $C_1$ . Next suppose  $\alpha' \in C_0 \setminus \{\alpha\}$ . Then the only 5-subset containing  $\alpha \cap \alpha'$  and at distance three from  $\alpha$  is  $(\alpha \cap \alpha') \cup (X \setminus \alpha \cup \alpha')$ , and this is fixed setwise by  $G_{\alpha \cap \alpha'}$ ; since  $G_{\alpha \cap \alpha'}$  is maximal in  $G$  it follows that these 5-subsets form a  $G$ -orbit  $C_3$  of size 55. Finally, a 5-subset  $\beta$  contained in  $X \setminus \alpha$  has stabiliser  $D_{10}$ , and we deduce that all 5-subsets contained in the complement of a block of  $C_0$  form a  $G$ -orbit of length 66 which must be  $C_2$ . The order of the stabiliser of a 5-subset in  $C_1, C_2$  and  $C_3$  is 2, 12 and 10 respectively so, by Theorem 4.1, none of these orbits can be completely transitive with  $\delta \geq 3$ . The covering radius of  $C_0$  is two, not three, and so  $C_0$  is also not completely transitive. (Presumably none of these orbits is even completely regular, this is certainly the case for  $C_0$  [??].)

Now consider case (ii). If  $G = M_{12}$ , then there are two orbits on 6-subsets, we obtain a completely transitive design, but again in this case we have  $\delta = 2$ . Thus  $G = M_{11}$ . By [2], the inner product of the permutation characters for the actions of  $M_{12}$  on the cosets of this subgroup  $G$  and on the blocks of the  $5 - (12, 6, 1)$  Steiner system is two, whence  $G$  has two orbits,  $C_0$  and  $C_3$  say, on the set  $B$  of 132 blocks of the  $5 - (12, 6, 1)$  Steiner system. These orbits have lengths 22 and 110, the block stabilisers being  $A_6$  and  $3^2 : [2^4]$  respectively.

Since both of these subgroups are transitive on  $\alpha \times (X \setminus \alpha)$ , where  $\alpha \in B$  is fixed by the subgroup, it follows that  $G$  has two orbits on the 6-subsets not in  $B$ , namely  $C_1$  of length 132, and  $C_2$  of length 660. Moreover,  $C_0$  is a completely transitive design, as is its opposite. Now  $C_0$  is the set of blocks of the  $3 - (12, 6, 2)$  design preserved by  $G$ ; since each 3-subset is contained in two blocks of  $C_0$  it follows that  $\delta = 3$ . A computer calculation shows that the dual degree of this design is two, whence its covering radius is at most two. (In fact  $C_0$  is completely regular, but  $C_2$  is the union of two orbits of  $M_{12}$ .) [Thanks to Bill Martin for the computations.]

From Table 1 in [7] we find that  $M_{22}$  has at least  $k$  orbits on  $k$ -sets when  $k \geq 6$ . Diagram 3 of [7] gives detailed information about the action of  $M_{22}$  on subsets of  $X$ , and this provides information about  $\text{Aut}(M_{22})$  as well. For any orbit of  $\text{Aut}(M_{22})$  is either an orbit of  $M_{22}$ , or the union of two  $M_{22}$  orbits of the same length. From Diagram 3 we see that if  $k \geq 6$ , there are three  $M_{22}$ -orbits of different lengths which are pairwise adjacent in  $J(22, k)$ . Thus no completely transitive designs arise in connection with  $M_{22}$  or  $\text{Aut}(M_{22})$ . From Diagram 2 of [7] we see that if  $8 \leq k \leq 12$  then there are three orbits of  $M_{23}$  on  $k$ -sets, pairwise at distance one in  $J(23, k)$ . So  $k \leq 7$  in this case. From Diagram 1 of the same source, we see that  $M_{24}$  has three pairwise adjacent orbits on 12-subsets, whence we have  $k \leq 11$ .

In cases (i) and (ii) we get an example:  $C_0$  is the set of blocks of the Steiner system. Similarly in case (iv) with  $k = 7$  and case (v) with  $k = 8$  the blocks of the Steiner system give examples. The Witt design on 23 points is completely transitive. [\*\*\* We need to consider the orbits of  $M_{23}$  on 5- and 6-subsets; these may be completely transitive, but have  $\delta < 3$ ? For  $M_{24}$  we have to consider the set of all  $k$ -subsets which contain a block of the Witt design on 24 points. By [7],  $M_{24}$  has three orbits on  $k$ -subsets when  $8 \leq k \leq 11$ . These have  $\delta < 3$  as well.\*\*\*]

**Higman Sims and the Conway group  $Co_3$ :** For  $G = HS$ ,  $v = 176$  and  $k(176 - k)$  divides  $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ . From the inequality  $\frac{1}{k-1} \binom{176}{k} \leq |G|$  it follows that  $k \leq 19$ , and then the divisibility condition implies that  $k$  is 8, 11 or 16. These need to be analysed. [\*\*\* Mmm \*\*\*]

For  $G = Co_3$ , it follows from Corollary 3.4 that  $k \leq 6$ , and from the fact that  $k(276 - k)$  divides  $|G|$ , we have  $k = 6$ .

The remaining infinite families of 2-transitive groups are the following:

- (i) The projective groups:  $L_d(q) \leq G \leq P\Gamma L_d(q)$  where  $d \geq 3$  and  $v = (q^d - 1)/(q - 1)$ , or  $G = A_7 < L_4(2)$  with  $v = 15$ .
- (ii) The affine groups:  $G = N.G_0 \leq A\Gamma L_d(q)$  where  $d \geq 1$  and  $v = q^d$ . Here  $N$  is the group of translations acting regularly on the vector space  $X = F + q^d$ , and  $G_0 \leq \Gamma L_d(q)$  acts transitively on the non-zero vectors.
- (iii) The Symplectic groups:  $G = Sp_{2m}(2)$  acting 2-transitively on the set of  $v = 2^{2m-1} + \varepsilon 2^{m-1}$  nondegenerate quadratic forms of type  $\varepsilon = \pm$ , which polarise to the symplectic form preserved by  $G$ .

We do not treat the symplectic groups at all. We make a partial analysis of the other two cases.

**Projective groups:**  $L_d(q) \leq G \leq P\Gamma L_d(q)$  where  $d \geq 3$  and  $v = (q^d - 1)/(q - 1)$ , or  $G = A_7 < L_4(2)$  with  $v = 15$ .

There are two constraints on  $k$  which must be satisfied if  $G$  is completely transitive. First, since  $G \leq P\Gamma L_d(q)$ , we have

$$|P\Gamma L_d(q)| \geq \frac{1}{k-1} \binom{v}{k}. \quad (4.1)$$

This provides an upper bound on  $k$ . To establish a lower bound, we first show that  $\alpha$  cannot be an  $i$ -dimensional subspace, for  $1 \leq i \leq d - 2$ .

The size of the symmetric difference of any two  $i$ -dimensional subspaces is at least  $2q^i$  and thus the minimum distance in  $J(v, k)$  between two  $i$ -dimensional subspaces is  $q^i$ . By Theorem 4.2(b) we may assume that  $k \geq 5$ , so  $q^{i+1} - 1 \geq 5(q - 1)$  and therefore  $q^i \geq 4$ . It follows that if we delete any two points  $p$  and  $p'$  from  $\alpha$  and replace them by two points  $q$  and  $q'$  not in  $\alpha$ , the resulting  $k$ -set  $\beta$  is not a subspace. Hence  $\beta \notin C_0 \cup C_1$ , whence it must lie in  $C_2$ . Now if  $q'$  is not in the span of  $\beta \cup q$  then  $\beta$  spans an  $(i + 2)$ -dimensional subspace, otherwise it spans an  $(i + 1)$ -dimensional subspace. If  $i \leq d - 3$  then both cases may occur, it follows that  $G$  has at least two orbits on  $C_2$ , and is not completely transitive.

Suppose then that  $i = d - 2$ , that is,  $\alpha$  is a hyperplane. Let  $p$  and  $p'$  be distinct points from  $\alpha$ . Let  $q$  and  $r$  be two points not in  $\alpha$  such that the unique line through  $q$  and  $r$  meets  $\alpha$  in a point distinct from  $p$  and  $p'$ , and let  $q'$  be a point on the line through  $p$  and  $q$  which is distinct from  $p$  and  $q$ . (So  $q' \notin \alpha$ .) Suppose

$$\beta := \alpha \setminus \{p, p'\} \cup \{q, r\} \quad \beta' := \alpha \setminus \{p, p'\} \cup \{q, q'\}.$$

Arguing as in the previous paragraph, we see that both  $\beta$  and  $\beta'$  lie in  $C_2$ . Suppose that there is an element  $g$  of  $G$  such that  $\beta^g = \beta'$ .

If  $\alpha^g$  is not equal to  $\alpha$  then we have  $|\alpha \cap \alpha'| \geq k-4$ , but  $\alpha \cap \alpha'$  is a subspace and therefore has size at most  $k - q^{d-2}$ . So  $q^{d-2} \leq 4$ , that is either  $G = L_4(2)$  and  $k = 7$  or, since  $k \geq 5$ , we have  $G \geq L_3(4)$  and  $k = 5$ . In both cases  $|\alpha \cap \alpha^g| = k-4$  and  $\delta = 4$ , and so  $\alpha^g$  contains  $\{q, q'\}$  and  $\alpha \cap \alpha^g \subseteq \alpha \setminus \{p, p'\}$ . In the case where  $G \geq L_3(4)$  this means that the line containing  $\{q, q'\}$  is  $\alpha^g$  and it meets  $\alpha$  in a point different from  $x$ , which is a contradiction. Similarly in the case  $G = L_4(2)$  the line  $\{p, q, q'\}$  must lie in the subspace  $\alpha'$  which is not the case. Hence  $\alpha^g = \alpha$ , and so  $(\alpha \setminus \{x, x'\})^g \subseteq \alpha \cap \beta'$  and consequently  $(\alpha \setminus \{p, p'\})^g = \alpha \setminus \{p, p'\}$ . Therefore  $\{p, q, q'\}^{g^{-1}} = \{p^{g^{-1}}, q, r\}$  is a collinear triple, which is a contradiction since  $p^{g^{-1}} \in \{p, p'\}$ . Thus we have shown that  $\alpha$  spans the whole space  $X$ .

If  $\alpha$  is not a subspace it follows that there is a line which meets  $\alpha$  in at least two points, but is not contained in it. Suppose  $p$  is a point in  $\alpha$  and  $\ell$  is a line on  $p$  which meets  $\alpha$  in exactly  $x$  points, where  $2 \leq x \leq q$ . By Theorem 4.1 we know that  $G_{\alpha, p}$  is transitive on those lines through  $p$  which are not contained in  $\alpha$ . Hence every line on  $p$  meets  $\alpha$  in at least  $x$  points. It follows that the number of lines on  $p$  is a lower bound on  $k-1$ , and thus we have

$$k-1 \geq \frac{v-1}{q}. \quad (4.2)$$

We are now going to apply this to the inequality in (4.1). If  $p$  is prime and  $q = p^a$  then

$$|P\Gamma L(d, q)| = \frac{a}{q-1} (q^d - 1) \cdots (q^d - q^{d-1}) < q^{d^2}. \quad (4.3)$$

We need to compare this with the ratio  $\binom{v}{k}/(k-1)$ . A routine calculation shows that when  $v \geq 2k$ , this is an increasing function of  $k$ . Since  $v/k < (v-i)/(k-i)$  when  $0 < i < k$  we also have

$$\binom{v}{k} > \left(\frac{v}{k}\right)^k.$$

From (4.2) we see that  $k \geq v/q$ , whence  $v/k \geq q$  and  $k \geq q^{d-2} + q^{d-3}$ . (Note that  $d \geq 3$ .) Accordingly (4.3) implies that if  $G$  is completely transitive then

$$q^{d^2+d-1} \geq (k-1)q^{d^2} > q^{q^{d-2}+q^{d-3}},$$

implying in turn that

$$d^2 + d - 1 > q^{d-2} + q^{d-3}. \quad (4.4)$$

This yields the following possibilities:

$$d = 3, q \leq 9; \quad d = 4, q \leq 3; \quad 5 \leq d \leq 7, q = 2.$$

Using (4.2) with  $k = \lfloor (v-1)/q \rfloor + 1$  in place of (4.4) the above list reduces to:

$$d = 3, q \leq 5; \quad d = 4, q = 2.$$

[\*\*\* So here are six more cases; I know how to eliminate the  $q = 2$  ones \*\*\*]

**Affine groups:**  $G = N.G_0 \leq \text{AGL}_d(q)$  where  $d \geq 2$ ,  $v = q^d$ ,  $N$  is the group of translations and  $G_0 \leq \text{GL}_d(q)$  is transitive on non-zero vectors.

We aim to show that if  $G$  is completely transitive then  $k \geq q^{d-1}$ . Suppose  $\alpha \in C_0$  and  $\alpha$  is an affine subspace with dimension  $i$ , where  $i \leq d-2$ . The size of the symmetric difference of any two  $i$ -dimensional subspaces is at least  $2(q-1)q^{i-1}$ . By Theorem 4.2(b) we have that  $q^i = k \geq 5$  and so any two  $i$ -dimensional subspaces are at distance at least four in  $J(v, k)$ . As in the projective case, it follows that  $C_2$  splits into at least two orbits under the action of  $G$ . Thus if  $\alpha$  is a subspace then it has dimension  $d-1$  and  $k = q^{d-1}$ .

Suppose then that  $\alpha$  is not a subspace and let  $p$  be a point in  $\alpha$ . Since  $G_\alpha$  is transitive on  $\alpha$ , there must be a line on  $p$  which contains both a point in  $\alpha \setminus p$  and a point not in  $\alpha$ . As  $G_\alpha$  is transitive on the set of lines through  $p$  which contain a point not in  $\alpha$ , it follows that every line through  $p$  contains a point of  $\alpha \setminus p$ . Therefore

$$|\alpha \setminus p| = k - 1 \geq \frac{v-1}{q-1} = \frac{q^d-1}{q-1} > q^{d-1}.$$

Thus we have shown that  $k \geq q^{d-1}$ .

If  $p$  is prime and  $q = p^a$  then

$$|\text{AGL}(d, q)| = aq^d |\text{GL}(d, q)| \leq q^{d^2+d+1}$$

while

$$\binom{v}{k} > \binom{v}{q^{d-1}} > q^{q^{d-1}}.$$



Consequently, if  $G$  is completely transitive, we must have

$$q^{(q+1)^2} \geq (k-1)q^{d^2+d+1} \geq \binom{v}{k} > q^{q^{d-1}}$$

and thus

$$(d+1)^2 > q^{d-1}.$$

This leaves the possibilities

$$d = 2, q \leq 8; \quad d = 3, q \leq 3; \quad 4 \leq d \leq 6.$$

[ \*\*\* Presumably some of these can be eliminated by using the exact values for  $|AGL(d, q)|$  and  $\binom{v}{k}/(k-1)$ ; I have not done these computations yet. \*\*\*]

## References

- [1] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*. Springer-Verlag, Berlin (1989).
- [2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
- [3] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Reports Suppl.* **10** (1973).
- [4] C. D. Godsil, *Algebraic Combinatorics*. Chapman and Hall, New York (1993).
- [5] Chris D. Godsil and John Shawe-Taylor, Distance-regularised graphs are distance-regular or distance biregular, *J. Combinatorial Theory, Series B*, **43** (1987) 14–24.
- [6] W. M. Kantor, Homogeneous designs and geometric lattices, *J. Combinatorial Theory, Series A* **38** (1985), 66–74.
- [7] E. S. Kramer, S. S. Magliveras and D. M. Mesner,  $t$ -Designs from the large Mathieu groups, *Discrete math.* **36** (1981), 171–189.
- [8] W. J. Martin, *Completely Regular Subsets*. Ph. D. Thesis, University of Waterloo (1992).

- [9] W. J. Martin, Completely regular designs of strength one, *J. Algebraic Combin.* **3** (1994), 177–185.
- [10] W. J. Martin, Completely regular designs, *J. Combin. Des.* **6** (1998), 261–273.
- [11] A. D. Meyerowitz, Cycle-balanced partitions in distance-regular graphs, *J. Combin. Inform. System Sci.* **17** (1992), 39–42.